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1986 J. Phys. A: Math. Gen. 19 L69

(http://iopscience.iop.org/0305-4470/19/2/009)

J. Phys. A: Math. Gen. 19 (1986) L69–L75. Printed in Great Britain

LETTER TO THE EDITOR

Phase transitions in two-dimensional stochastic cellular automata

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Received 1 August 1985, in final form 15 October 1985

Abstract. Two-dimensional stochastic cellular automata with nearest-neighbour couplings are investigated. Depending on the rules, low-level noise phases are classified into ferro, glassy, roll, glassy-roll, and antiferro, glassy-antiferro, antiferro-roll, glassy-antiferro-roll, 'coexistent phase of ferro and antiferro', 'labyrinth' and 'turbulence'. Also, the oscillating phases with period 2 for the above phases are observed in other rules. Transformations from ferro to antiferro and from fixed point to periodic patterns are shown. The nature of phase transitions due to a change of noise is investigated.

Phase transitions are common phenomena both in equilibrium and non-equilibrium systems. In equilibrium statistical mechanics, Ising models have played important roles. In non-equilibrium systems, the transition phenomena are more abundant (Nicolis and Prigogine 1977, Haken 1978), though a simple general model with discrete states is not available. In this letter, we consider a class of stochastic cellular automata (SCA), in order to study the general aspects of phase transitions both for equilibrium and non-equilibrium systems. The model might be regarded as a generalisation of Ising models and may also be a typical discrete model for non-equilibrium transitions.

Recently, Wolfram (1983, 1984) and Packard and Wolfram (1985) have investigated one- and two-dimensional cellular automata (CA) from the viewpoint of dynamical systems theory. A phase transition is observed in one-dimensional CA if the rules are changed at some lattice points (Grassberger *et al* 1984, Kinzel 1985), while it is found in the coupled map lattices as the coupling parameter is changed (Kaneko 1984, 1985a, Aizawa *et al* 1985). In the present letter a class of two-dimensional stochastic cellular automata with nearest-neighbour couplings is investigated on a square lattice with a periodic boundary condition, mainly focusing on the transition phenomena caused by a change of noise strength.

Inclusion of noise in cellular automata is important for the following reasons: first, noise plays the role of temperature in equilibrium systems. Thus, a phase transition as the noise level changes is expected for a system with a dimension higher than one. Second, deterministic cellular automata can have a huge number of attractors. Inclusion of noise brings about a jump among attractors and leads to the selection of a small number of physical states (Kauffman 1969, Kaneko 1985b). Lastly noise plays an important role for the formation of patterns in non-equilibrium systems.

The rule for the evolution for the SCA in the present letter is given by

$$s_{i,j}^{t+1} = \begin{cases} I(n_{i,j}^t, s_{i,j}^t) & \text{with probability } 1-p \\ 1-I & \text{with probability } p \end{cases} \tag{1}$$

where $n_{i,j} = s_{i,j-1} + s_{i,j+1} + s_{i-1,j} + s_{i+1,j}$ with $s_{i,j}^t = 0$ or 1. The suffices i,j denote a lattice site while the superscript t shows the discrete time step. In the present letter the system size is chosen to be 32×32. The 'rule' I is a function which takes the value 0 or 1. Noise is added so that $s_{i,j}$ changes its values with probability p. Since the value of I has two choices for each possible 5×2 states ($n = 0, 1, 2, 3, 4$ and $s = 0, 1$), the number of possible rules is 2^{10}.

We have investigated 2^{10} rules for low-level noises ($p \sim 10^{-4}$). The essential feature, however, can be seen in the following symmetric rules.

The symmetric rule is defined as the one which has symmetry about the transformation of 0 and 1. This gives the condition $I(n, s) = 1 - I(4-n, 1-s)$. Hereafter we restrict ourselves to the symmetric rules, which have 2^5 choices.

Furthermore, the number of independent rules is reduced by the method of sublattices. The first transformation is the ferro-antiferro transformation (FAF trsf). Let us consider the sublattice $s_{i,j}^{\mathrm{o}}$ with $i+j = \text{odd}$ and $s_{i,j}^{\mathrm{e}}$ with $i+j = \text{even}$. If we apply the transformation $s_{i,j}'^{\mathrm{o}} = 1 - s_{i,j}^{\mathrm{o}}$ and $s_{i,j}'^{\mathrm{e}} = s_{i,j}^{\mathrm{e}}$ for the system with a rule $\bar{I}(n, s') = I(4-n, s')$, it is shown that the dynamics is equivalent to the system for the rule $I(n, s)$. Thus, the result for the rule $\bar{I}(n, s)$ is automatically obtained from that of the rule $I(n, s)$ by the above transformation. The transformation changes a pattern

$$\begin{matrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{matrix} \quad \text{into} \quad \begin{matrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{matrix}$$

and may be called a FAF trsf (ferro-antiferro transformation).

Another transformation is obtained by the use of a temporal sublattice. If the transformation $s_{i,j}'^{2t} = s_{i,j}^{2t}$ and $s_{i,j}'^{2t+1} = 1 - s_{i,j}^{2t+1}$ is applied to a system with a rule $\bar{I}(n, s') = I(4-n, 1-s')$, it is shown that the dynamics are equivalent to a system with a rule $I(n, s)$. The transformation changes a time series 1 1 1 1 1 1 (fixed point) to 1 0 1 0 1 0 (period-2 oscillation) and may be called a periodic-fixed point transformation (PFP trsf).

Taking the above two transformations into account, the number of independent symmetric rules is reduced to 10, which will be studied in the following, where a rule is represented by a code $(i_0 i_1 i_2 i_3 i_4)$, where $i_j = I(j, 0)$ $(I(j, 1) = 1 - I(4-j, 0)$ from symmetry) or by a rule number defined by $\Sigma_{k=0}^{4} 2^k i_k$.

Patterns at low-level noises are shown in table 1 for rules from 0 to 15. The pattern 'ferro' means a phase with a long range order with broken symmetry about 0 and 1, i.e. the ferromagnetic phase in usual Ising systems. The magnetisation $m = \Sigma_{i,j}(2s_{i,j} - 1)$ appears for $p < p_c$, where p_c is the 'transition noise level'. In the 'glassy' phase there appears a short range order for a weak noise, i.e. the correlation $c_{k,1} = \langle (2s_{i+k,j+1} - 1)(2s_{i,j} - 1) \rangle$ is not small if neither k nor j is large, though the long range order cannot be attained even if the noise level p goes to zero (see figure 1). No phase transition appears as p goes to zero. Since the relaxation becomes slower and slower as $p \to 0$ and the dynamics include a topological constraint, then here we tentatively

Table 1. Patterns of elementary SCA; rule numbers, their codes and patterns at low-level noise, and the presence or absence of long range order (LRO) at low noise are shown. If a rule is derived from another rule number which gives a simpler pattern (i.e. fixed point or ferro is simpler than periodic or antiferro (AF)) by FAF or PFP transformation, the type of transformation and the primary rule are written as 'transformation number'. If a rule is invariant against the FAF transformation, it is written as self-dual (SD).

Rules 0, 1, 2, 3, 4, 5, 10, 17, 18, 19 are primary in the sense that they have neither 'periodic' nor 'AF' phases. Other rules are obtained by FAF or PFP transformation from the above rules.

Number	Code	Pattern	LRO	Transformation
0	0 0 0 0 0	Trivial	No	
1	0 0 0 0 1	Glassy	No	
2	0 0 0 1 0	Ferro	Yes	
3	0 0 0 1 1	Ferro	Yes	
4	0 0 1 0 0	Coexistence of ferro and AF	?	(SD)
5	0 0 1 0 1	Ferro	Yes	
6	0 0 1 1 0	Periodic-AF-roll	Yes	PFP * FAF(19)
7	0 0 1 1 1	Periodic-AF	Yes	PFP * FAF(3)
8	0 1 0 0 0	AF	Yes	FAF(2)
9	0 1 0 0 1	Glassy-AF-roll	No	FAF(18)
10	0 1 0 1 0	⟨Additive turbulence⟩	No	(SD)
11	0 1 0 1 1	Periodic-AF	Yes	PFP * FAF(5)
12	0 1 1 0 0	Periodic-roll	Yes	PFP(19)
13	0 1 1 0 1	Periodic-glassy-roll	No	PFP(18)
14	0 1 1 1 0	Periodic-⟨labyrinth⟩	No	PFP(17) (SD)
15	0 1 1 1 1	Periodic-glassy-AF	No	PFP * FAF(1)

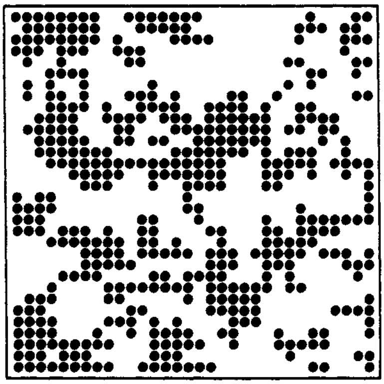

Figure 3. Overlap function $C(T)$ for the rule 0 0 0 0 1 for $p = 0.001$. The data can be fitted well by $a_1 e^{-T/\tau_1} + a_2 e^{-T/\tau_2}$ with $a_1 = 0.37$, $\tau_1 = 3.2 \times 10^2$, $a_2 = 0.63$ and $\tau_2 = 1.4 \times 10^3$.

call the phase 'glassy'. The 'roll' phase is shown in figure 2(b), which is the pattern with

$$\begin{matrix} 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0. \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \end{matrix}$$

A glassy-roll pattern is a roll phase only with a short range order.

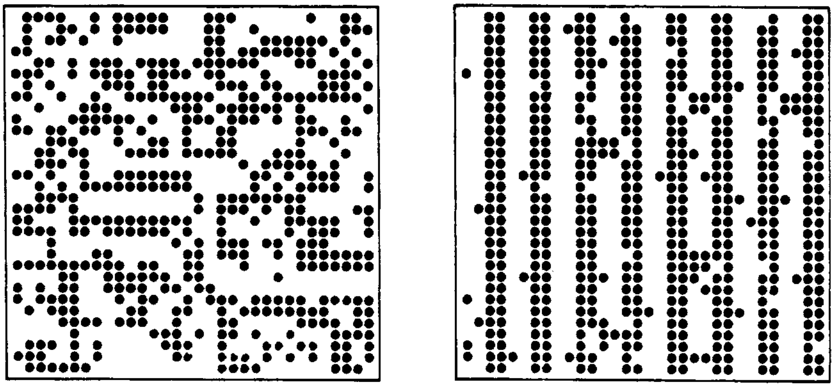

Figure 2. Snapshot of SCA with the rule 1 0 0 1 (roll) for $p=0.0045$ (a) from random initial configurations (after 10^4 steps), (b) from ordered initial configurations (after 10^4 steps).

'Periodic' means a pattern which repeats 1 and 0 in time at each site (i.e. the period is 2). All periodic patterns are obtained by PFP trsf. For example, the ordered phase for the rule 1 0 0 1 1 is roll. Applying PFP trsf, the rule is changed to 0 1 1 0 0, which gives the periodic-roll pattern as shown in table 1. Since the rule with a number $n(n>15)$ is obtained by the PFP trsf from the rule with a number $31-n$, phases for the rules with $n(n>15)$ are omitted from the table.

FAF trsf is also useful. If we apply the transformation to a rule for which the low-level noise phase is a pattern such as ferro, glassy, roll or glassy-roll, the low-level noise phase for the transformed rule is the corresponding pattern AF

$$\left(\text{antiferro};\begin{matrix}1&0&1&0\\0&1&0&1\\1&0&1&0\end{matrix}\right),\ \text{glassy-AF},\ \text{AF-roll}\begin{pmatrix}1&1&0&0\\0&0&1&1\\1&1&0&0\end{pmatrix},$$

or glassy-AF-roll respectively. For example, if we apply FAF trsf and PFP trsf successively to the rule 0 1 1 0 1 (periodic-glassy-roll), rule 0 1 0 0 1 is obtained, which yields glassy-AF-roll for low-level noise.

The rules 0 0 0 0 0 (0), 0 0 1 0 0 (4), 0 1 0 1 0 (10), 0 1 1 1 0 (14) (and those obtained by PFP trsf from these rules) are invariant under FAF trsf. Except for the trivial rule 0 (which preserves the initial condition), these rules show interesting behaviour for low-level noise, which will be discussed next.

Here some typical patterns in table 1 are discussed in a little more detail, though the complete accounts will be reportted elsewhere (Akutsu *et al* 1985).

(a) Glassy (rule 0 0 0 0 1). As the noise level p is lowered, the relaxation becomes slower and slower. We calculated the overlap function

$$C(T)=(1/N^2)\sum_{ij}\langle(2s_{i,j}^{t_0+T}-1)(2s_{i,j}^{t_0}-1)\rangle \tag{2}$$

where $\langle\ \rangle$ indicates long time average and $N=32$, after transients have decayed out ($t_0\sim10^3$). $C(T)$ cannot be fitted by a single exponential function and it seems to be represented by $a_1\mathrm{e}^{-T/\tau_1}+a_2\mathrm{e}^{-T/\tau_2}+\cdots$ (see figure 3). The fastest relaxation τ_1 is estimated as follows: the probability that the neighbouring sites take the same values (0 0 or 1 1) is defined as q. For small noise p, the self-consistent approximation for q

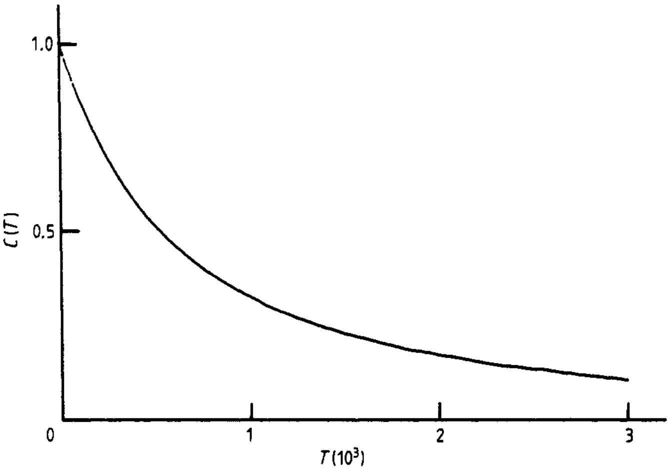

Figure 3. Overlap function $C(T)$ for the rule 0 0 0 0 1 for $p = 0.001$. The data can be fitted well by $a_1 e^{-T/\tau_1} + a_2 e^{-T/\tau_2}$ with $a_1 = 0.37$, $\tau_1 = 3.2 \times 10^2$, $a_2 = 0.63$ and $\tau_2 = 1.4 \times 10^3$.

gives the equation $(1-q)^4 = 8p$, from which the relaxation time τ_1 is given by $(3p)^{-1}$. The numerical results confirm the above estimation. The detailed explanation for the form of $C(T)$, however, remains a problem for the future.

(b) Roll (rule 1 0 0 1 1). The order-disorder transition occurs as can be seen in figure 3. The transition is first order in the sense that it has a hysteresis. If the noise level is lowered, the ordered state appears at $p = p_- \sim 0.0044$, while it remains up to $p = p_+ \sim 0.0049$ when the noise level is increased on this ordered phase. The roll magnetisation is defined in a similar way to the staggered magnetisation for the antiferromagnetic order, which shows a large jump at $p \sim p_+$ or p_-.

Here we note that this ordered state breaks the spatial symmetry (rotational symmetry with angle $\pi/2$), since the rule itself is isotropic.

(c) Ferro-antiferro coexistence (rule 0 0 1 0 0). Both the ferro and AF patterns are stable for low-level noise for the rule. If we start from the random initial configurations, some systems fall into ferro and some into AF, while the others show the coexistence of ferro and antiferro clusters even after 4×10^4 steps, and fixed patterns are not attained (see figure 4).

(d) 'Labyrinth' (rule 1 0 0 0 1). The rule 1 0 0 0 1 (0 1 1 1 0 for the periodic case) gives a pattern in figure 5 for low-level noise. Both the roll

$$\begin{matrix} 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \end{matrix} \quad \text{and a single roll} \quad \begin{matrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{matrix}$$

are not destroyed by a change of $s_{i,j}$ at a single site. These two types of rolls form a labyrinth structure only with a short range order. The relaxation of overlap function (2) is quite slow for low-level noise, which is fit by a single exponential decay

(e) 'Additive turbulence' (rule 0 1 0 1 0 or 1 0 1 0 1). The rule 0 1 0 1 0 shows a chaotic behaviour. No local order with periodic or fixed patterns has been observed (see figure

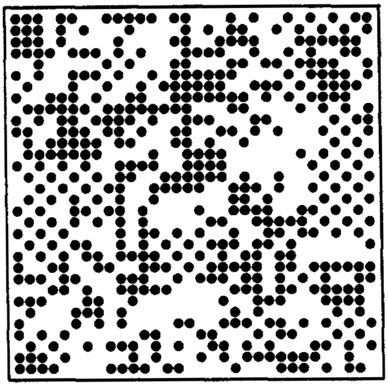

Figure 4. Snapshot of SCA with the rule 0 0 1 0 0 (coexistence of ferro and AF) for $p = 0.001$.

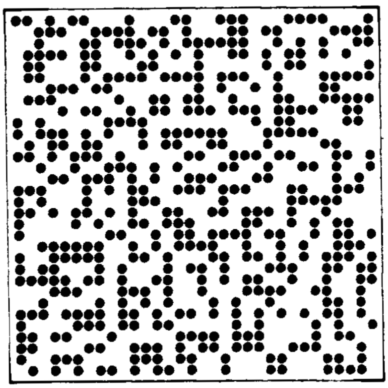

Figure 5. Snapshot of SCA with the rule 1 0 0 0 1 (labyrinth) for $p = 0.001$.

6). Simple patterns such as ferro, AF, periodic-roll and single-roll can exist as a special solution for $p = 0$, but they are easily destroyed by a change of $s_{i,j}$ at a single site.

The above rules are additive in the sense of Wolfram (1983), since they are written as

$$s_{i,j}^{n+1} = s_{i+1,j}^{n} + s_{i-1,j}^{n} + s_{i,j+1}^{n} + s_{i,j-1}^{n} + s_{i,j}^{n} \qquad (\text{mod } 2) \qquad (\text{rule } 0\,1\,0\,1\,0)$$

or

$$s_{i+1,j}^{n} + s_{i-1,j}^{n} + s_{i,j+1}^{n} + s_{i,j-1}^{n} + s_{i,j-1}^{n} \qquad (\text{mod } 2) \qquad (\text{rule } 1\,0\,1\,0\,1).$$

Thus, the apparently turbulent behaviour for the above rules can be understood by the superposition of the behaviour obtained by a single site excitation, which is analogous to that by rule 90 of one-dimensional elementary CA (Wolfram 1983).

We have clarified various ordered phases for elementary stochastic cellular automata. Though some of the rules show behaviour common to the usual Ising models (such as ferro or AF), other remarkable phases have been observed. For the 'glassy' phase, it is not yet certain whether the phase is similar to the low temperature phase for the spin glass model with short range interactions. Recently, Fredrickson and Andersen (1984) have considered Ising spin systems with kinetics with some constraint

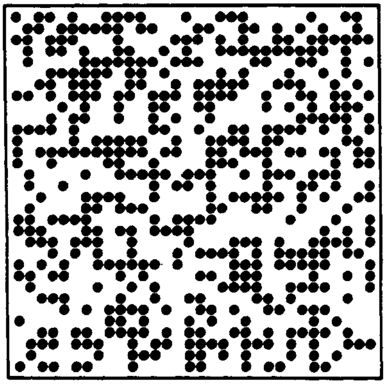

Figure 6. Snapshot of SCA with the rule 0 1 0 1 0 (turbulence) for $p = 0.001$.

and expected a glass transition. The rule 0 0 0 0 1 imposes a topological constraint on the dynamics. Monte Carlo simulation or SCA with unusual kinetics may be of relevance for the understanding of some aspects of glassy behaviour.

The periodic patterns may be related to some non-equilibrium phase transitions in an ensemble of oscillator systems (Kuramoto 1981). Roll patterns and their first order transitions may be related to the dislocation in roll patterns in Benard systems (Ahlers and Behringer 1978, Gollub and Steinman 1981, Fauve *et al* 1984).

Stochastic cellular automata are simple and show a variety of new phenomena, which may open a curtain on a new era of phase transition study.

The authors would like to thank Mr Yukito Iba for stimulating discussions and critical comments.

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